Linear Circuit Analysis: Ch 11, AC Steady State Power

Tatpong Katanyukul

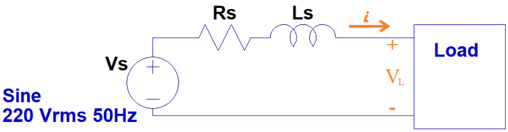
มหาวิทยาลัยขอนแก่น
KHON KAEN UNIVERSITY

5 เมษายน พ.ศ. 2564

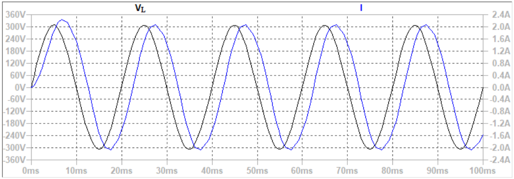
Key Concepts:

- Instantaneous power
- Average power
- Effective or rms values
- Complex power
- Power factor
- Power factor correction

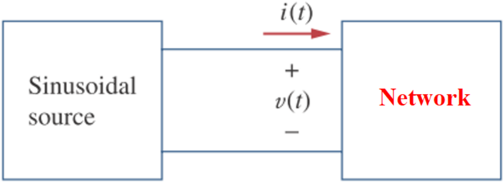
Recall an AC Steady-State System



- $V_L = i \cdot Z_{\text{Load}}$



Instantaneous Power

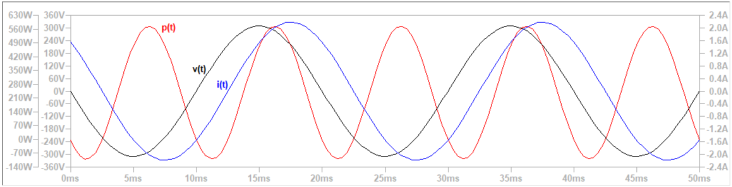


Instantaneous power:

- Instantaneous power is a power at a particular point in time.
- $p(t) = v(t) \cdot i(t)$
  where $v(t)$ and $i(t)$ are voltage and current at the particular time.
- When $p(t) > 0$, the network is consuming power.
- When $p(t) < 0$, the network in supplying power.

Instantaneous Power of an AC System (I)

$$p(t) = v(t) \cdot i(t)$$



Given $v(t) = V_m \cos(\omega t + \theta_V)$ and $i(t) = I_m \cos(\omega t + \theta_I)$,

$$\begin{aligned} p(t) &= V_m \cos(\omega t + \theta_V) \cdot I_m \cos(\omega t + \theta_I) \\ &= V_m I_m \cos(\omega t + \theta_V) \cdot \cos(\omega t + \theta_I). \end{aligned}$$

Instantaneous Power of an AC System (II)

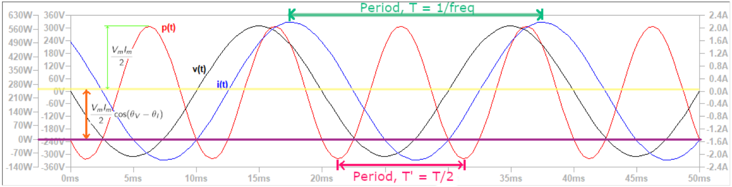
$$p(t) = V_m I_m \cos(\omega t + \theta_V) \cdot \cos(\omega t + \theta_I).$$

From a product of cosine identity:
$\cos(A)\cos(B) = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$, therefore

$$\begin{aligned} p(t) &= \frac{V_m I_m}{2}\{\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V + \theta_I)\} \\ &= \underbrace{\frac{V_m I_m}{2}\cos(\theta_V - \theta_I)}_{\text{time-constant part}} + \underbrace{\frac{V_m I_m}{2}\cos(2\omega t + \theta_V + \theta_I)}_{\text{time-varying part}}. \end{aligned} \quad (1)$$

Instantaneous Power of an AC System (III)

$$p(t) = \underbrace{\frac{V_m I_m}{2}\cos(\theta_V - \theta_I)}_{\text{offset}} + \underbrace{\frac{V_m I_m}{2}\cos(2\omega t + \theta_V + \theta_I)}_{\text{sinusoid}}.$$



- $\frac{V_m I_m}{2}\cos(\theta_V - \theta_I)$ (constant to time) plays an offset role.
- $\frac{V_m I_m}{2}\cos(2\omega t + \theta_V + \theta_I)$ is a sinusoidal part with its frequency doubled from the original.

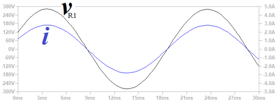
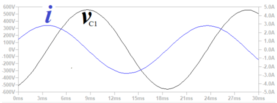
  $\Rightarrow$ its magnitude is $\frac{V_m I_m}{2}$.

Recall Properties of R, L, and C

- R: $\mathbb{I} = \mathbb{V}/R = \frac{V_m}{R}\angle\theta_V$
  $\Rightarrow I$ and V are in phase: $\theta_V = \theta_I$.
- L: $\mathbb{I} = \mathbb{V}/(j\omega L) == \frac{V_m}{\omega L}\angle(\theta_V - \frac{\pi}{2})$
  $\Rightarrow I$ lags V by $\frac{\pi}{2}$: $\theta_V - \theta_I = \frac{\pi}{2}$.
- C: $\mathbb{I} = \mathbb{V} \cdot (j\omega C) == V_m\omega C\angle(\theta_V + \frac{\pi}{2})$
  $\Rightarrow I$ leads V by $\frac{\pi}{2}$: $\theta_V - \theta_I = -\frac{\pi}{2}$.

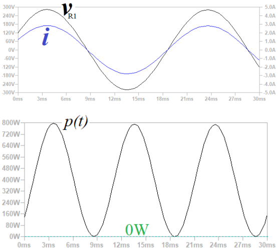
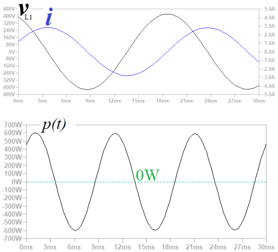
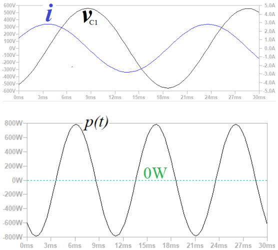
Instantaneous Power on R, L, and C (I)

$$p(t) = \underbrace{\frac{V_m I_m}{2}\cos(\theta_V - \theta_I)}_{\text{offset}} + \underbrace{\frac{V_m I_m}{2}\cos(2\omega t + \theta_V + \theta_I)}_{\text{sinusoid}}.$$

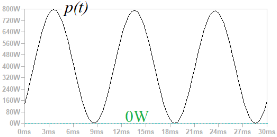
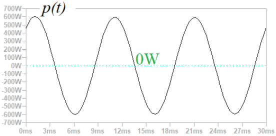
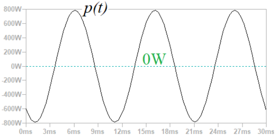
R: In phase	L: Lagging by $\frac{\pi}{2}$	C: Leading by $\frac{\pi}{2}$
		
$\theta_V = \theta_I$ $\Rightarrow$ offset $= \frac{V_m I_m}{2}$.	$\theta_V - \theta_I = \frac{\pi}{2}$ $\Rightarrow$ offset $= 0$.	$\theta_V - \theta_I = -\frac{\pi}{2}$ $\Rightarrow$ offset $= 0$.
With this offset, $p_R \geq 0$ all the times.	With no offset, sinusoid will oscillate around 0.	

Instantaneous Power on R, L, and C (II)

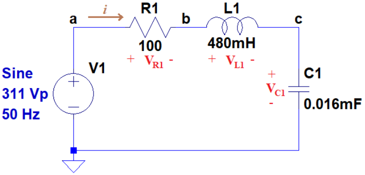
$$p(t) \quad = \quad \frac{V_m I_m}{2} \cos(\theta_V - \theta_I) + \frac{V_m I_m}{2} \cos(2\omega t + \theta_V + \theta_I).$$

R: In phase	L: Lagging	C: Leading
		
offset $= \frac{V_m I_m}{2}$. $p(t) \geq 0$ always.	offset $= 0$. $p(t) \geq 0$ half the times $p(t) \leq 0$ another half	

Instantaneous Power on R, L, and C (III)

<table>
<tr><td>R: In phase

</td><td>L: Lagging

</td><td>C: Leading

</td></tr>
<tr><td>$p(t) \geq 0$ always.</td><td colspan="2">$p(t) \geq 0$ half the times
$p(t) \leq 0$ another half</td></tr>
<tr><td>R always consumes power.</td><td colspan="2">L and C are energy-storage devices.
$p(t) > 0 \Rightarrow$ stores power
$p(t) < 0 \Rightarrow$ releases power</td></tr>
</table>

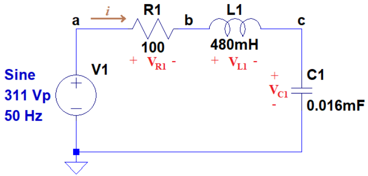
Example: Instantaneous Power (I)



- $V_1(t) = 311\sin(314.2t)$
- $\mathbb{V}_1 \equiv 311\angle - \frac{\pi}{2}$
- $\mathbb{Z}_{R1} = 100\Omega$
- $\mathbb{Z}_{L1} = j150.8\Omega$
- $\mathbb{Z}_{C1} = -j198.9\Omega$
- $\mathbb{Z} = 100 - j48.1\Omega$

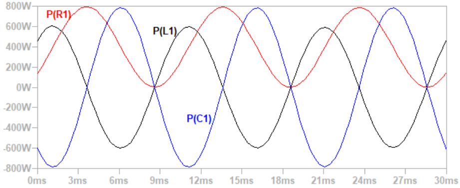
- $\therefore \mathbb{I} = \mathbb{V}_1/\mathbb{Z} = 2.8\angle - 1.12$
- $i(t) = 2.8\cos(314.2t - 1.12)$
- $\mathbb{V}_{R1} = \mathbb{I}\cdot\mathbb{Z}_{R1} = 280.3\angle - 1.12$
- $\mathbb{V}_{L1} = \mathbb{I}\cdot\mathbb{Z}_{L1} = 422.6\angle 0.45$
- $\mathbb{V}_{C1} = \mathbb{I}\cdot\mathbb{Z}_{C1} = 557.4\angle - 2.69$
- $V_{R1}(t) = 280.3\cos(314.2t - 1.12)$
- $V_{L1}(t) = 422.6\cos(314.2t + 0.45)$
- $V_{C1}(t) = 557.4\cos(314.2t - 2.69)$

Example: Instantaneous Power (II)



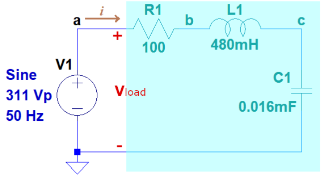
- $I_m = 2.8;\ \theta_I = -1.12$
- $V_m(R1) = 280.3;\ \theta_V(R1) = -1.12$
- $V_m(L1) = 422.6;\ \theta_V(L1) = 0.45$
- $V_m(C1) = 557.4;\ \theta_V(C1) = -2.69$
- $p_{R1}(t) = \frac{(280.3)(2.8)}{2} + \frac{(280.3)(2.8)}{2}\cos(2(314.2)t - 1.12 - 1.12)$
- $p_{L1}(t) = \frac{(422.6)(2.8)}{2}\cos(2(314.2)t + 0.45 - 1.12)$
- $p_{C1}(t) = \frac{(557.4)(2.8)}{2}\cos(2(314.2)t - 2.69 - 1.12)$

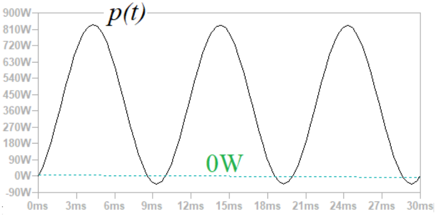
Example: Instantaneous Power (III)



- $p_{R1}(t) = 392.4 + 392.4\cos(628.4t - 2.24)$
- $p_{L1}(t) = 591.6\cos(628.4t - 0.67)$
- $p_{C1}(t) = 780.4\cos(628.4t - 3.81)$

Example: Instantaneous Power (IV)

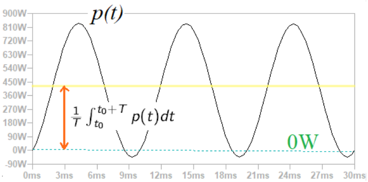




- $V_{\text{load}} = V_1$
  $\Rightarrow \therefore \mathbb{V}_{\text{load}} = 311\angle -\frac{\pi}{2}$
- $V_m(\text{load}) = 311;\ \theta_V(\text{load}) = -\frac{\pi}{2}$

$$\begin{aligned} p_{\text{load}}(t) &= \frac{(311)(2.8)}{2}\cos(-\tfrac{\pi}{2} + 1.12) + \frac{(311)(2.8)}{2}\cos(628.4t - \tfrac{\pi}{2} - 1.12) \\ &= 391.9 + 435.4\cos(628.4t - 2.69) \end{aligned}$$

- Instantaneous power is too detailed and it tells too little about overall power consumption.
- Average Power:
  $P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt$
  where T is a time period; t_0 is an arbitrary point in time.



Average Power (II)

- From $p(t) = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_V + \theta_I)$
- Thus, average power
  $P = \frac{1}{T}\int_{t_0}^{t_0+T}\left\{\frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_V + \theta_I)\right\}dt.$
- Since integration over a period of a periodic function is 0.
  $\int_{t_0}^{t_0+T} f(t)dt = 0$ when T is a period of $f(t)$,
- then
  $P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + 0.$

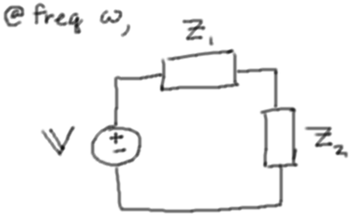
That is,

$$P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I). \tag{2}$$

Average Power (III)

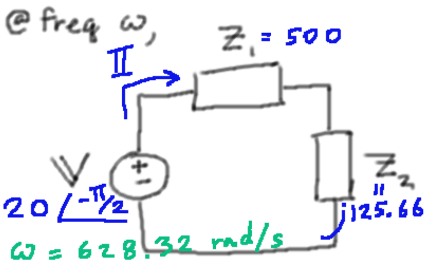
- Average power over R: $P_R = \frac{V_m I_m}{2}$
- Average power over (ideal) L: $P_L = 0$
- Average power over (ideal) C: $P_C = 0$
- Average power over an impedance $\mathbb{Z} = Z_m \angle \theta_z$: $P_Z = \frac{V_m I_m}{2} \cos \theta_z$
  Recall: $\mathbb{Z} = \mathbb{V}/\mathbb{I}$ and $\theta_Z = \theta_V - \theta_I$.

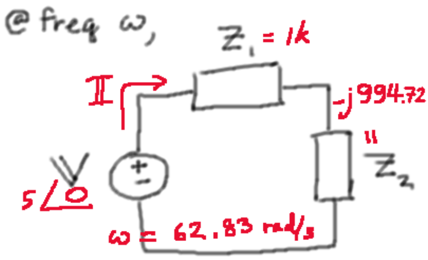
Example: Average Power (I)



- Find an average power over a combination of loads Z_1 and Z_2.
- (1) Let voltage source: 20 Vp sine 100 Hz;
  Z_1 is a 500Ω resistor and Z_2 is a 200mH inductor.
- (2) Let voltage source: 10 Vpp cosine 10 Hz; Z_1 is a 1kΩ resistor and Z_2 is a 16μF inductor.

Example: Average Power (II)





$\mathbb{I} = \mathbb{V}/(\mathbb{Z}_1 + \mathbb{Z}_2) = 0.0388\angle{-1.817}$.

$P = \frac{(20)(0.0388)}{2}\cos(-\frac{\pi}{2} + 1.817)$

$\therefore P = 0.3763$ W.

$\mathbb{I} = 3.54m\angle 0.783$.

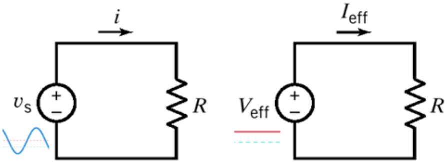
$P = \frac{(5)(3.54m)}{2}\cos(0 - 0.783)$

$\therefore P = 6.59$ mW.

Effective Values (I)

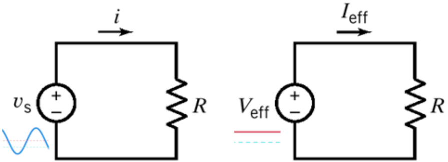
- An AC voltage can be described in many ways: peak value V_p, peak-to-peak value V_{pp}, magnitude $V_m = V_p$, or amplitude $V_a = V_m = V_p$.
- How are these quantities compared to a DC voltage?
  $\Rightarrow$ Rationale is to describe a magnitude of an AC voltage in a way to comprehend its real work.
- I.e., describing the magnitude of an AC voltage by an amount of a DC voltage that will deliver the same average power to a resistor.

Effective Values (II)



- V_{eff}: an amount of a dc voltage that delivers the same average power as its counterpart V_s.
- Average power on R by an AC voltage
  $P_{ac} = \frac{1}{T}\int_0^T \frac{v_s^2}{R} dt$
- Average power on R by a DC voltage
  $P_{eff} = \frac{v_{eff}^2}{R}$.

Effective Values (III)



Solve for v_{eff} (delivering the same power as v_s:

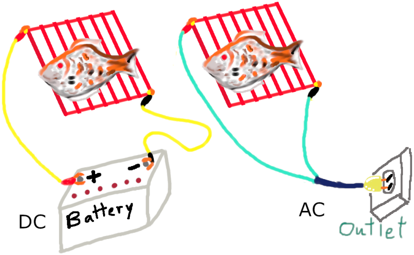
$$v_{eff} = \sqrt{\frac{1}{T}\int_0^T v_s^2 dt}. \tag{3}$$

- Since the result is obtained by squaring, averaging, and finding a square root, it is called a "Root-Mean-Square" value or "RMS" for short.

Effective Values (IV)

- The effective value of an ac voltage is the amount of its dc equivalence, i.e., supplying the same average power on a resistor.
- So is an effective value of an ac current.

$v_{rms} = \sqrt{\frac{1}{T}\int_0^T v_s^2 dt}$	$i_{rms} = \sqrt{\frac{1}{T}\int_0^T i_s^2 dt}$



Effective voltage is like finding a dc equivalence to cook the fish as if the fish is cooked by an ac.

RMS on Sinusoid

Given $v_s(t) = V_m \cos(\omega t)$,

$$\begin{aligned}
V_{rms} &= \sqrt{\frac{1}{T}\int_0^T (V_m \cos(\omega t))^2\, dt} \\
&= V_m \sqrt{\frac{1}{T}\int_0^T \cos^2(\omega t) dt} \\
&= V_m \sqrt{\frac{1}{T}\int_0^T \frac{1+\cos(2\omega t)}{2} dt} \\
&= V_m \sqrt{\frac{1}{2}\cdot \underbrace{\frac{1}{T}\int_0^T 1 dt}_{1} + \underbrace{\frac{1}{T}\int_0^T \frac{\cos(2\omega t)}{2} dt}_{0}} \\
&= \frac{V_m}{\sqrt{2}}. \qquad (4)
\end{aligned}$$

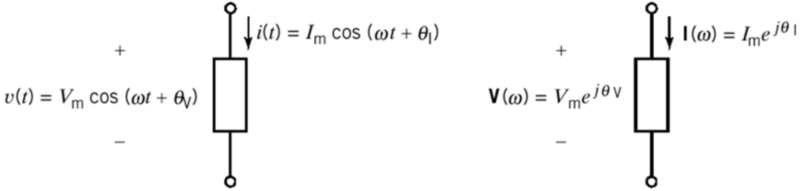
RMS Values

- $V_{rms} = \frac{V_m}{\sqrt{2}}$
- $I_{rms} = \frac{I_m}{\sqrt{2}}$

$\Rightarrow$ Conveniently, average power

$$\begin{aligned} P &= \frac{V_m \cdot I_m}{2} \cos(\theta_V - \theta_I) \\ &= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos(\theta_V - \theta_I) \\ &= V_{rms} \cdot I_{rms} \cos(\theta_V - \theta_I). \end{aligned}$$

Complex Power (I)



Time domain   Frequency domain

- voltage and current can be represented in frequency domain as complex numbers.
- So is power!

Complex Power (I)

Given $\mathbb{V} = V_m \angle \theta_V$ and $\mathbb{I} = I_m \angle \theta_I$, complex power delivered to the element is defined as:

$$\begin{aligned} \mathbb{S} = \frac{\mathbb{V} \cdot \mathbb{I}^*}{2} &= \frac{(V_m \angle \theta_V) \cdot (I_m \angle - \theta_I)}{2} \\ &= \frac{V_m \cdot I_m}{2} \angle (\theta_V - \theta_I) \\ &= V_{rms} \cdot I_{rms} \angle (\theta_V - \theta_I) \end{aligned} \tag{5}$$

where $\mathbb{I}^*$ is a complex conjugate of $\mathbb{I}$.

Complex Power (II)

- Complex power $\mathbb{S} = \frac{V_m \cdot I_m}{2} \angle(\theta_V - \theta_I)$.
- Magnitude $|\mathbb{S}| = \frac{V_m \cdot I_m}{2}$ is called "apparent power".
- Its rectangular form:

$$\mathbb{S} = \underbrace{\frac{V_m \cdot I_m}{2} \cos(\theta_V - \theta_I)}_{P} + j \underbrace{\frac{V_m \cdot I_m}{2} \sin(\theta_V - \theta_I)}_{Q}$$

$$\mathbb{S} = P + jQ.$$

- Its real part P is average power or real power.
- Its imaginary part Q is called "reactive power".
- $\mathbb{S}$'unit is VA (Volt-Amp).

  P's is W.

  Q's is VAR (Volt-Amp Reactive).

Power Factor

- Complex power $\mathbb{S} = \underbrace{\frac{V_m \cdot I_m}{2}\cos(\theta_V - \theta_I)}_{P} + j\underbrace{\frac{V_m \cdot I_m}{2}\sin(\theta_V - \theta_I)}_{Q}$.
- Real power $P = \underbrace{\frac{V_m \cdot I_m}{2}}_{|\mathbb{S}|} \cdot \underbrace{\cos(\theta_V - \theta_I)}_{\text{power factor}}$.
- Power factor $pf = \cos(\theta_V - \theta_I)$.
- pf angle $\theta_V - \theta_I > 0 \Rightarrow$ "lagging"
  Current lags voltage $\rightarrow$ inductive load.
- pf angle $\theta_V - \theta_I = 0 \Rightarrow$ "in-phase"
  Current and voltage are in-phase $\rightarrow$ purely resistive load.
- pf angle $\theta_V - \theta_I < 0 \Rightarrow$ "leading"
  Current leads voltage $\rightarrow$ capacitive load.

Example: Load Impedance (I)

An electric load operates at 240 Vrms. The load consumes an average power of 8 kW at a lagging power factor of 0.8.

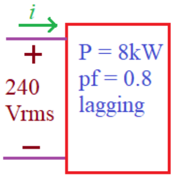
(a) Calculate the complex power of the load.

(b) Calculate the impedance of the load.

Solution for (a):

- $pf = \cos(\phi) = 0.8$ and $P = 8kW = |\mathbb{S}| \cdot \cos(\phi)$.
  $\Rightarrow |\mathbb{S}| = 8k/0.8 = 10$ kVA.
- lagging $\Rightarrow \phi > 0$.
  $\Rightarrow \phi = \cos^{-1} 0.8 = 0.6435$ rad.
- $\therefore Q = |\mathbb{S}| \cdot \sin(\phi) = 10k \sin(0.6435) = 6$ kVAR.
- $\mathbb{S} = P + jQ = 8 + j6$ kVA.

Example: Load Impedance (II)

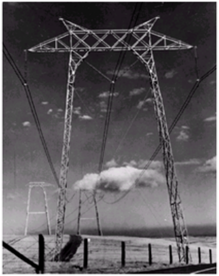


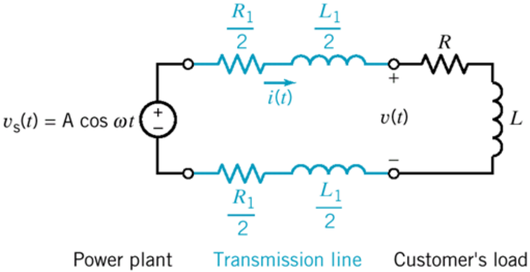
Solution for (b):

- $P = 8kW = \frac{V_m I_m}{2} \cdot pf = V_{rms} \cdot I_{rms} \cdot pf = 240 \cdot I_{rms} \cdot 0.8$
  $\Rightarrow I_{rms} = 41.67$ A.
- Impedance $\mathbb{Z} = \frac{\mathbb{V}}{\mathbb{I}} = \frac{V_m}{I_m}\angle(\theta_V - \theta_I) = \frac{V_{rms}}{I_{rms}}\angle\phi = \frac{240}{41.67}\angle 0.6435.$
  $\Rightarrow \mathbb{Z} = 4.608 + j3.456\ \Omega.$

PF and Transmission Loss (I)

The transmission lines for electrical power





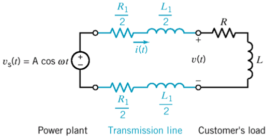
- Transmission lines are significantly different from ideal wires.

  $\mathbb{Z}_{line} = R_1 + j\omega L_1.$

- Average power losing on the lines

  $P_{line} = I_{rms}^2 \cdot Re\{\mathbb{Z}_{line}\} = I_{rms}^2 R_1.$

PF and Transmission Loss (II)

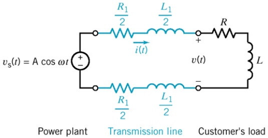


Average power losing on the lines

$$P_{line} = I_{rms}^2 \cdot Re\{\mathbb{Z}_{line}\} = I_{rms}^2 R_1.$$

- A large load is often inductive, e.g., machine, motor, pump, etc.
- Power consumed by the load is what its owner pays.

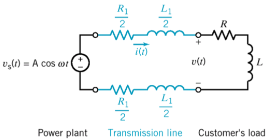
  $P_{load} = V_{rms} I_{rms} \cdot pf.$
- Power losing on the lines is just a waste of energy, which customers do not pay, but it costs an electric supplier.
- Given the same P_{load} and operating voltage V_{rms}, a lower pf causes a higher current I_{rms}. And, a higher current $I_{rms} \Rightarrow$ a higher P_{line}.



$$P_{load} = V_{rms} I_{rms} \cdot pf.$$
$$P_{line} = I_{rms}^2 \cdot Re\{\mathbb{Z}_{line}\} = I_{rms}^2 R_1.$$

Example: a 1.4kW Load with a lagging power factor of 0.8, operated at 220 Vrms 50Hz.

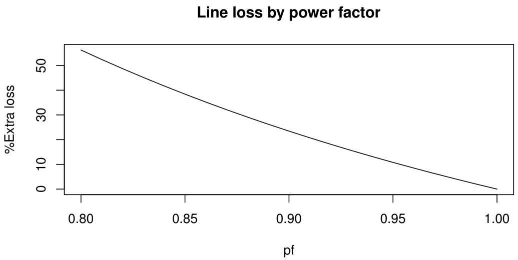
- Current i drawn by the load: $I_{rms} = \frac{1.4k}{(220)(0.8)} = 7.95$ A.
- Supposed $R_1 = 1\ \Omega$, $P_{line} = (7.95)^2(1) = 63.27$ W.
  $\Rightarrow$ To put it in perspective, if the load is run 20 hr/day, 350 days a year, that counts 7000 hours. Supposed the price is 5 baht/unit, energy loss: $E_{loss} = (63.27)(7000) = 442890$ Wh $= 442.89$ units.
  Thus, estimated loss is 2214.45 baht for this one customer.



$$I_{rms} = \frac{P_{load}}{V_{rms} \cdot pf}.$$
$$P_{line} = I_{rms}^2 R_1.$$

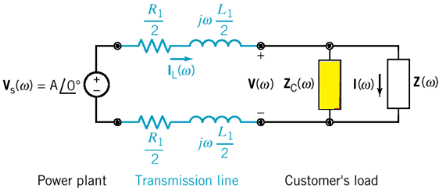
Line loss as a function of power factor

$$P_{line} = \left(\frac{P_{load}}{V_{rms} \cdot pf}\right)^2 R_1.$$



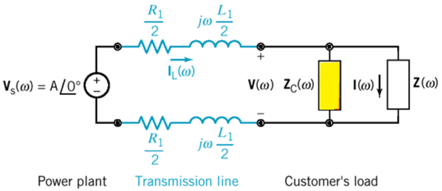
With pf of 0.8, the power line loss is over 50% of the ideal case ($pf = 1$).

Power Factor Correction (I)



- To mitigate, a customer is required to have the power factor above an agreeable level.
- To "correct" power factor of a load, a correcting impedance is installed across the terminals of the customer's load.
  $\Rightarrow$ having the correcting impedance parallel to the load retains the load operating voltage.

Power Factor Correction (II)

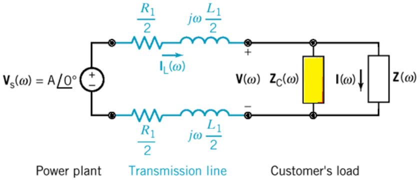


The correcting impedance should turn power factor of the load to

$$pfc = \cos \phi_p,$$

where pfc and ϕ_p are the target power factor and the target pf angle.

Given load impedance $\mathbb{Z} = R + jX$ and correcting impedance $\mathbb{Z}_c = R_c + jX_c$, the correcting impedance should not consume power itself: $R_c = 0 \Rightarrow \mathbb{Z}_c = jX_c$. (It must be purely reactive, C or L.)



Target power factor: $pfc = \cos\phi_p$.

Original load: $\mathbb{Z} = R + jX$.

Correcting impedance: $\mathbb{Z}_c = jX_c$.

- corrected impedance: $\mathbb{Z}_p = \frac{\mathbb{Z}\mathbb{Z}_c}{\mathbb{Z}+\mathbb{Z}_c} = R_p + jX_p = Z_p\angle\theta_p$.
- note: pf angle = load angle, i.e., $\theta_p = \phi_p$.

$$\Rightarrow \mathbb{Z}_p = Z_p\angle\phi_p$$

$$\Rightarrow \phi_p = \tan^{-1}\frac{X_p}{R_p} \Rightarrow pfc = \cos(\tan^{-1}\frac{X_p}{R_p}) \Rightarrow \frac{X_p}{R_p} = \tan(\cos^{-1} pfc).$$

Power Factor Correction (IV)

(1) $\mathbb{Z}_p = \frac{\mathbb{Z}\mathbb{Z}_c}{\mathbb{Z}+\mathbb{Z}_c} = R_p + jX_p$.

(2) $\frac{X_p}{R_p} = \tan(\cos^{-1} pfc)$.

Target: $pfc = \cos\phi_p$.

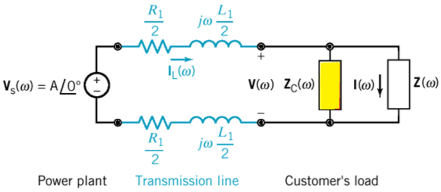
Impedances: $\mathbb{Z} = R + jX$ and $\mathbb{Z}_c = jX_c$.

Work the math, from (1):

$$\mathbb{Z}_p = \frac{(R+jX)(jX_c)}{(R+jX)+jX_c} = \underbrace{\frac{RX_c^2}{R^2+(X+X_c)^2}}_{R_p} + j\underbrace{\frac{R^2X_c+(X_c+X)XX_c}{R^2+(X+X_c)^2}}_{X_p}.$$

Put the result into (2):

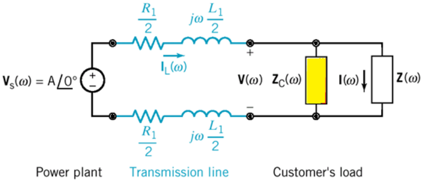
$$\frac{R^2X_c+(X_c+X)XX_c}{RX_c^2} = \tan(\cos^{-1} pfc).$$



Solve for X_c in terms of R, X, and pfc:

$$X_c = \frac{R^2 + X^2}{R \tan(\cos^{-1} pfc) - X}. \tag{6}$$

Example: Power Factor Correction



Reactive value:

$$X_c = \frac{R^2+X^2}{R\tan(\cos^{-1} pfc)-X}$$

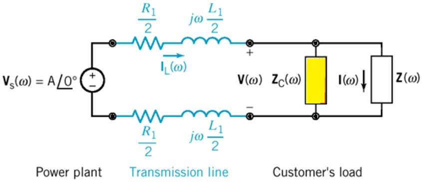
Example. A 1.4kW load with a lagging pf of 0.8, operating at 220 Vrms 50 Hz, is required to be corrected for pf of 0.95.

Solution:

Recalling from the previous example, $I_{rms} = 7.95$ A. Then,

$$\mathbb{Z} = \frac{220}{7.95}\angle(\underbrace{+}_{lagging}\cos^{-1}0.8) = 27.67\angle 0.64 = 22.19 + j16.52\ \Omega.$$

Thus, $X_c = \frac{22.19^2+16.52^2}{22.19\cdot\tan(\cos^{-1}0.95)-16.52} = -82.9$. Since $X_c < 0$, the correcting impedance must be capacitive: $X_c = -\frac{1}{\omega C}$. Given $\omega = 2\pi(50) = 314.16$ rad/s, $C = 38.4\ \mu$F (or larger).



Reactive value:

$$X_c = \frac{R^2 + X^2}{R\tan(\cos^{-1} pfc) - X}$$

- The larger *pfc* (closer to 1), the better.
- $pfc \to 1 \Rightarrow \cos^{-1} pfc \to 0 \Rightarrow \tan(\cos^{-1} pfc) \to 0$.
- Sign of X_c is opposite to the sign of X.
  - $\Rightarrow$ Correct capacitive load with inductor.
  - $\Rightarrow$ Correct inductive load with capacitor.

Power Factor Correction: Inductive Load (II)

Typical loads are inductive. Thus, $X_c = -\frac{1}{\omega C}$ and

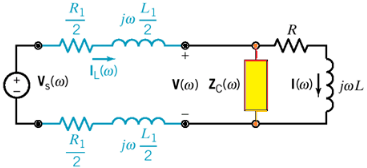
$$\begin{aligned} -\frac{1}{\omega C} &= \frac{R^2 + X^2}{R\tan(\cos^{-1} pfc) - X} \\ \omega C &= \frac{X - R\tan(\cos^{-1} pfc)}{R^2 + X^2} \\ &= \frac{R}{R^2 + X^2} \cdot \left(\frac{X}{R} - \tan(\cos^{-1} pfc)\right). \end{aligned}$$

Given an original pf angle $\phi = \tan^{-1}\frac{X}{R}$, hence

$$C = \frac{R}{\omega \cdot (R^2 + X^2)} \cdot (\tan\phi - \tan\phi_c), \tag{7}$$

where $\phi = \cos^{-1} pf$ and $\phi_c = \cos^{-1} pfc$.

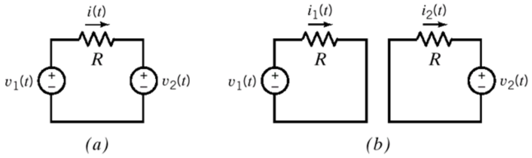
Example: Power Factor Correction on Inductive Load



Given the original load $\mathbb{Z} = 100 + j100\ \Omega$ at 60 Hz, find C to improve pf to 0.95.

<u>Solution</u>:

- $\phi = \tan^{-1}\frac{X}{R} = 0.785$ rad.
- $\phi_c = \cos^{-1} pfc = 0.318$ rad.
- $\omega = 2\pi f = 377$ rad/s.
- $C = \frac{100}{377 \cdot (100^2 + 100^2)} \cdot (\tan 0.785 - \tan 0.318) = 8.9\ \mu$F (or larger).

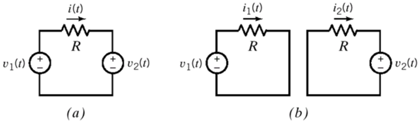


- Superposition: $i = i_1 + i_2$.
- Instantaneous power: $p = i^2 R = (i_1 + i_2)^2 R = (i_1^2 + i_2^2 + 2 i_1 i_2)R$.

Average power:

$$
\begin{aligned}
P &= \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T (i_1^2 + i_2^2 + 2 i_1 i_2) R dt \\
&= P_1 + P_2 + \frac{2R}{T}\int_0^T (i_1 \cdot i_2) dt.
\end{aligned}
$$

Power Superposition for Multi-Frequency Excitation (II)



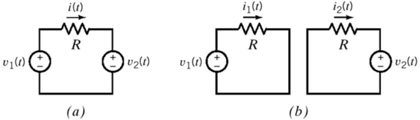
Average power:

$$P = P_1 + P_2 + \frac{2R}{T}\underbrace{\int_0^T (i_1 \cdot i_2)dt}$$

Given $i_1 = I_1 \cos(\omega_1 t + \theta_1)$ and $i_2 = I_2 \cos(\omega_2 t + \theta_2)$,
$\int_0^T (i_1 \cdot i_2)dt = I_1 I_2 \int_0^T \cos(\omega_1 t + \theta_1)\cos(\omega_2 t + \theta_2)dt.$

From $\cos A \cos B = \frac{\cos(A-B)+\cos(A+B)}{2}$,

$$\begin{aligned}\int_0^T (i_1 \cdot i_2)dt &= \frac{I_1 I_2}{2}\int_0^T (\cos((\omega_1 - \omega_2)t + \theta_1 - \theta_2))\,dt \\ &+ \frac{I_1 I_2}{2}\underbrace{\int_0^T (+\cos((\omega_1 + \omega_2)t + \theta_1 + \theta_2))\,dt}_{0}.\end{aligned}$$



Average power:

$$P = P_1 + P_2 + \frac{2R}{T} \underbrace{\int_0^T (i_1 \cdot i_2) dt}$$

$$\begin{aligned} \int_0^T (i_1 \cdot i_2) dt &= \frac{I_1 I_2}{2} \int_0^T \left(\cos((\omega_1 - \omega_2)t + \theta_1 - \theta_2)\right) dt \\ &= \begin{cases} 0 & \text{for } \omega_1 \neq \omega_2, \\ \frac{I_1 I_2 T}{2} \cos(\theta_1 - \theta_2) & \text{for } \omega_1 = \omega_2. \end{cases} \end{aligned}$$

That is,

$$P = P_1 + P_2 + \begin{cases} 0 & \text{for } \omega_1 \neq \omega_2, \\ I_1 I_2 R \cos(\theta_1 - \theta_2) & \text{for } \omega_1 = \omega_2. \end{cases} \tag{8}$$

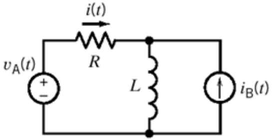
"The average power delivered to a circuit by several sinusoidal sources, acting together, is equal to the sum of the average power delivered to the circuit by each source acting alone, if and only if no two of the sources have the same frequency."

$$\begin{aligned} P_{total} &= P_1 + P_2 + \cdots + P_N \\ &= \sum_i P_i, \end{aligned}$$

where P_i is a power computed as if the i^{th} sinusoidal source acting alone and no two sources have the same frequency.

Caution! Superposition principle cannot be applied to power in general.

Example: Power Superposition (I)

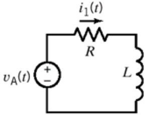


Find power P_R consumed by R in this setting: $v_A(t) = 155.6\cos(377t)$; $i_B(t) = 1.2\cos(314.16t)$; $R = 50\ \Omega$ and $L = 0.5$ H.

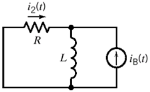
Solution:

- Both sources have different frequencies.
  $\Rightarrow$ use superposition to work on one source at a time.
- Since both sources are sinusoidal and have different frequencies, power superposition can be applied.
  $P_R = P_R(v_A) + P_R(i_B)$.

Example: Power Superposition (II)

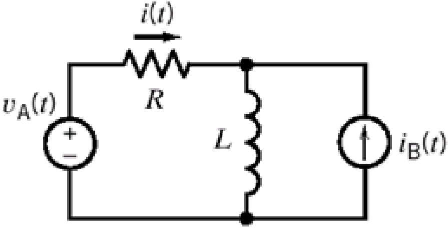


- $\omega_1 = 377$ rad/s
- $Z_R = 50;\ Z_L = j188.5.$
- $\mathbb{V}_A = 155.6\angle 0.$
- $\mathbb{I}_1 = \frac{\mathbb{V}_A}{50+j188.5}$
  $= 0.798\angle -1.31$
- $P_R(v_A) = I_{1rms}^2 R$
  $= \left(\frac{0.798}{\sqrt{2}}\right)^2 50 = 15.92$ W.



- $\omega_2 = 314.16$ rad/s
- $Z_R = 50;\ Z_L = j157.08.$
- $\mathbb{I}_B = 1.2\angle 0.$
- $\mathbb{I}_2 = -\mathbb{I}_B \cdot \frac{\mathbb{Y}_R}{\mathbb{Y}_R+\mathbb{Y}_L}$
  $= -1.2\frac{(1/50)}{(1/50)+(1/j157.08)}$
  $= 1.143\angle -2.83.$
- $P_R(i_B) = I_{2rms}^2 R$
  $= \left(\frac{1.143}{\sqrt{2}}\right)^2 50 = 32.69$ W.

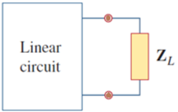
Example: Power Superposition (III)



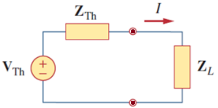
Since both sources have different frequency, the power superposition is valid:

$$
\begin{aligned}
P_R &= P_R(v_A) + P_R(i_B) \\
&= 15.92 + 32.69 = 48.61 W.
\end{aligned}
$$

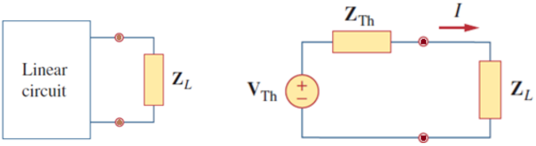
Maximum Average Power Transfer (I)



- At some situation, we may want to find a load Z_L such that the power delivered to the load is maximum.



- To simplify the task, the portal circuit is modeled by Thevenin equivalent circuit.
- Supposed $\mathbb{Z}_{Th} = R_{Th} + jX_{Th}$ and $\mathbb{Z}_L = R_L + jX_L$, current $\mathbb{I} = \frac{\mathbb{V}_{Th}}{\mathbb{Z}_{Th}+\mathbb{Z}_L}$.
- Average power $P_R = \frac{V_m I_m}{2} = \frac{1}{2} I_m^2 R_L = \frac{1}{2} \frac{|\mathbb{V}_{Th}|^2 R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}$
- Find X_L and R_L maximizing P_R.



$$P_R = \frac{1}{2}\frac{|\mathbb{V}_{Th}|^2 R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}.$$

- Notice that $X_L = -X_{Th}$ maximizing P_R.
- Solve $\frac{\partial P_R}{\partial R_L} = 0$ for R_L
  $\Rightarrow R_L = R_{Th}$.

Thus,

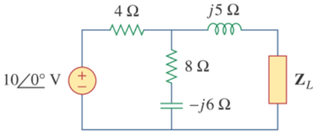
$$\mathbb{Z}_L = R_{Th} - jX_{Th} = \mathbb{Z}^*_{Th}. \quad (9)$$

The maximum power is:

$$P_{\max} = \frac{|\mathbb{V}_{Th}|^2_{Th}}{8R_{Th}}. \quad (10)$$

Example: Maximum Average Power Transfer (I)

Determine the load impedance $\mathbb{Z}_L$ maximizing the average power drawn from the circuit. What is the maximum average power?



Solution:

- Firstly, find Thevenin equivalent circuit.

  $\Rightarrow$ (V divider) $\mathbb{V}_{oc} = 10 \cdot \frac{8-j6}{4+8-j6} = 7.454\angle - 0.18.$

  $\Rightarrow$ (Mesh analysis)

  $-10 + 4\mathbb{I}_1 + (8 - j6)(\mathbb{I}_1 - \mathbb{I}_2) = 0.$

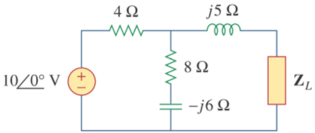
  $(8 - j6)(\mathbb{I}_2 - \mathbb{I}_1) + j5\mathbb{I}_2 = 0.$

  Solve for $\mathbb{I}_2 = \mathbb{I}_{sc} = 1.395\angle - 1.1696.$

  $\Rightarrow \mathbb{V}_{Th} = \mathbb{V}_{oc} = 7.454\angle - 0.18.$

  $\Rightarrow Z_{Th} = \mathbb{V}_{oc}/\mathbb{I}_{sc} = 2.93 + j4.47.$

Example: Maximum Average Power Transfer (II)



$$\mathbb{V}_{Th} = 7.454\angle - 0.18.$$

$$Z_{Th} = 2.93 + j4.47.$$

- $\mathbb{Z}_L = \mathbb{Z}^*_{Th} = 2.93 - j4.47\ \Omega.$
- $P_{\max} = \frac{|\mathbb{V}_{Th}|^2_{Th}}{8R_{Th}} = \frac{7.454^2}{8(2.93)} = 2.37\ \mathrm{W}.$

Further Study

David E. Johnson, Johnny R. Johnson, John L. Hilburn, and Peter D. Scott, Electric Circuit Analysis. Wiley 3rd edition (January 15, 1997).